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ON THE APPROXIMATE REPRESENTATION OF A FUNCTION OF TWO VARIABLES

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BY

E. L. MICKELSON

The purpose of this paper is to exhibit extensions of some of the known existence theorems on the approximate representation of a function of one variable to corresponding theorems for a function of two variables and somewhat to investigate approximate representation by means of surface spherical harmonics.

For convenience the paper is divided into two sections. The material of the first section extends to a function of two variables some of the work of Professor Jackson on the approximate representation of a function of one variable given in the first chapter of his Ithaca Colloquium Lectures.† Trigonometric approximation based on an extension to two variables of Jackson's approximating integral is the foundation upon which other forms of representation are built by means of cosine transformations. These transformations are in part responsible for the essential difference between this paper and a mere rephrasing of Jackson's work. The section is concerned only with a real continuous function of two variables; its extension to a function of any finite number of variables is apparent. Moreover, it is limited to bare essentials: for simplicity, only periods of 2π and intervals of length 2 are considered; material of a superficial nature obtained by generalizing the condition of Lipschitz is omitted; no attempt is made to find small values for the absolute constants which enter—the order of approximation alone is sought; some applications of the theory paralleling those of Jackson‡ to Fourier, to Legendre, and to the corresponding mixed approximations have been omitted at this time and reserved for further extension.

The discussion in the second section is confined to the representation of a real function on the surface of a unit sphere by partial sums of Laplace's series and certain other sums of surface spherical harmonics and to the convergence of the approximating sum of surface spherical harmonics which minimizes the surface integral of a power of the absolute error. An expression for an upper bound to the absolute error in the representation by a partial sum of La-

* Presented to the Society at Chicago, March 29, 1929; received by the editors in December, 1930.

† D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, New York, 1930; cited below as *Colloquium*.

‡ See *Colloquium*, pp. 18–32.

place's series is given in terms of the order of the sum; the expression is obtained from considerations of the mean value of the given function used in conjunction with the results of a simple lemma. The discussion of the convergence of the approximating sums in the sense of integrals parallels that of Jackson for trigonometric sums.

I. Trigonometric, polynomial, and mixed approximation

1. **The forms of the approximating functions.** The approximating functions to be used in this section are finite sums, the forms of which are herein defined and listed for reference. Let m and n be a pair of positive integers, and let (a_{ij}), (b_{ij}), (c_{ij}), and (d_{ij}), where i and j range independently over the integers from zero to m and from zero to n respectively, be sets of real constants.

By a trigonometric sum of order *at most* m in x and n in y is meant a sum of the form

$$T_{mn}(x, y) = \sum \begin{Bmatrix} a_{ij} \cos ix \cos jy + b_{ij} \cos ix \sin jy \\ + c_{ij} \sin ix \cos jy + d_{ij} \sin ix \sin jy \end{Bmatrix}; \tag{1}$$

by a polynomial of degree at most m in x and n in y is meant a sum of the form

$$P_{mn}(x, y) = \sum (a_{ij} x^i y^j); \tag{2}$$

by a mixed sum of order at most m in x and degree at most n in y is meant one of the form

$$H_{mn}(x, y) = \sum [(a_{ij} \cos ix + b_{ij} \sin ix) y^j]. \tag{3}$$

2. **Trigonometric approximating functions for two variables.** Let $g(x, y)$ be a continuous periodic function of period 2π in x and in y separately. Let m and n be two positive integers; let p and q be integers such that $2p-2 \leqq m \leqq 2p$ and $2q-2 \leqq n \leqq 2q$; let $I_{pq}(x, y)$ be defined by the integral

$$I_{pq}(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(x + 2u, y + 2v) F_{pq}(u, v) du dv, \tag{4}$$

where

$$F_{pq}(u, v) = \left[\frac{(\sin pu)(\sin qv)}{(p \sin u)(q \sin v)} \right]^4 \tag{5}$$

and

$$1/h_{pq} = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} F_{pq}(u, v) du dv, \text{ a positive constant.} \tag{6}$$

The integral (4) is the extension of Jackson's approximating function* to fit the case of two variables. By an argument entirely analogous to that used for one variable it follows that $I_{pq}(x, y)$ is a trigonometric sum of type (1) of order at most m in x and n in y; hence, when m and n are specified it and sums similar to it have the form of the desired approximating sum for the function $g(x, y)$.

From (6) it follows that

$$(7)\qquad \begin{aligned} &| g(x, y) - I_{pq}(x, y) | \\ &\leqq h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} | g(x + 2u, y + 2v) - g(x, y) | F_{pq}(u, v)dudv, \end{aligned}$$

so that the expression on the right furnishes an upper bound for the absolute value of the error in representing $g(x, y)$ by $I_{pq}(x, y)$. Evaluation of the integral on the right depends upon finding an expression for the absolute difference in the integrand through suitable restrictions on $g(x, y)$.

In anticipation of the results the following facts are noted for future reference†:

$$(8)\qquad \int_0^{\pi/2} (z \sin^4 pz)/(p^4 \sin^4 z)dz \leqq c_1'(1/p^2) \leqq c_1(1/m^2)$$

and

$$(9)\quad c_3(1/m) \leqq c_3'(1/p) \leqq \int_0^{\pi/2} (\sin^4 pz)/(p^4 \sin^4 z)dz \leqq c_2'(1/p) \leqq c_2(1/m),$$

where c_1, c_2, and c_3 are absolute positive constants.

3. **The modulus of continuity.** A suitable expression for the absolute difference in the integrand of (7) is obtained from considerations of the modulus of continuity of $g(x, y)$. It seems to be advisable at this point to lay a foundation for the remainder of this section by giving essential definitions and properties of the modulus of continuity together with demonstrations of the more involved facts.

Let $g(x, y)$ be continuous in a closed rectangular region R of the xy-plane. Define $\omega(\delta)$ to be the maximum of the absolute difference $|g(x_1, y_1) - g(x_2, y_2)|$ for all points (x_1, y_1), (x_2, y_2) in R for which $(x_1 - x_2)^2 + (y_1 - y_2)^2 \leqq \delta^2$. The

* See *Colloquium*, p. 3; also, for the case of two variables, C. E. Wilder, *On the degree of approximation to discontinuous functions by trigonometric sums*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 345–361; p. 358.

† See *Colloquium*, p. 5.

function $\omega(\delta)$, called the *modulus of continuity* of $g(x, y)$ in R, exists and has the following properties*:

(10) $\omega(\delta)$ is a continuous function of δ; $\omega(0)=0$;
$\omega(\delta)>0$ when $\delta>0$ unless $g(x, y)$ is constant in R;

(11) $$\omega(\delta_1) \leqq \omega(\delta_2),\ \delta_1 \leqq \delta_2;$$

(12) $$\omega(k\delta) \leqq k\omega(\delta),\ k \text{ a positive integer};$$
$$\omega(k\delta) \leqq (k+1)\omega(\delta),\ k \text{ any positive number};$$
$$\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2);$$

(13) $$2\omega(\delta_1)/\delta_1 \geqq \omega(\delta_2)/\delta_2,\ 0<\delta_1 \leqq \delta_2.$$

In case $\omega(\delta)$ does not exceed a quantitiy of the form $\lambda\delta$, λ a positive constant, $g(x, y)$ is such that $|g(x_1, y_1)-g(x_2, y_2)| \leqq \lambda[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$ and is said to satisfy a Lipschitz condition† with parameter λ.

For present purposes the definitions are extended to include finite sets of functions. Let $g_0(x, y), g_1(x, y), g_2(x, y), \cdots, g_k(x, y)$—hereafter denoted by $\{g_k(x, y)\}$—be a finite set of functions continuous in R with moduli of continuity $\{\omega_k(\delta)\}$. Define $\Omega(\delta)$ to be the greatest of the quantities $\omega_k(\delta)$ for each value of δ; call it the *uniform modulus of continuity* for the set $\{g_k(x, y)\}$ in R. Quite obviously $\Omega(\delta)$ has the same properties (10), (11), (12), and (13) as the ordinary modulus $\omega(\delta)$. If the functions $\{g_k(x, y)\}$ satisfy Lipschitz conditions with parameters $\{\lambda_k\}$ in R they all evidently satisfy such conditions with a single parameter Λ which is the largest of the set $\{\lambda_k\}$; under such circumstances the set $\{g_k(x, y)\}$ will be said to satisfy a Lipschitz condition with parameter Λ.

The relation between the uniform modulus of continuity of the partial derivatives of specified order of a given function and the uniform modulus of continuity of the partial derivatives of the same order of its cosine transform is not only essential for the method of this paper, but it holds some interest of its own. Let $g(x, y)$ be continuous in the square region $-1 \leqq x, y \leqq 1$; let $G(\theta, \phi)=g(\cos\theta, \cos\phi)$. If the modulus of continuity of $g(x, y)$ in the region $-1 \leqq x, y \leqq 1$ is $\omega(\delta)$, $G(\theta, \phi)$ has in every finite region a modulus of continuity $w(\delta)$ such that $w(\delta) \leqq \omega(\delta)$; moreover, if $g(x, y)$ satisfies a Lipschitz condition with parameter λ in the region $-1 \leqq x, y \leqq 1$, G satisfies such a condition everywhere and with the same parameter. Furthermore, if the

* de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, pp. 7–8.

† de la Vallée Poussin, op. cit., p. 9.

kth-order partial derivatives of $g(x, y)$ with respect to x and y exist and have a uniform modulus of continuity $\Omega(\delta)$ in $-1 \leqq x, y \leqq 1$, the kth-order partial derivatives of $G(\theta, \phi)$ with respect to θ and ϕ will have a uniform modulus of continuity $W(\delta)$ everywhere; if these derivatives of $g(x, y)$ satisfy a Lipschitz condition with parameter Λ the corresponding derivatives of $G(\theta, \phi)$ will satisfy such a condition everywhere with some parameter L. The relation of $W(\delta)$ to $\Omega(\delta)$ and of L to Λ, in general, is not so simple as in the case $k=0$. Although for particular values of k the relations are often more simple than those offered below, the latter are sufficiently acceptable generally.

LEMMA I. *Let $g(x, y)$ together with all its partial derivatives of order $k \geqq 1$ and lower be continuous in the region $-1 \leqq x, y \leqq 1$; let M be the maximum of the absolute values of these derivatives in the region; let the uniform modulus of continuity of the partial derivatives of order k be $\Omega(\delta)$ for $0 \leqq \delta \leqq 2^{3/2}$, the maximum diameter of the region, and let the symbol $\Omega(\delta)$ denote the value $\Omega(2^{3/2})$ for $\delta > 2^{3/2}$. Then if $\Omega(2^{3/2}) \neq 0$, $G(\theta, \phi) = g(\cos\theta, \cos\phi)$ is a periodic function of period 2π in θ and in ϕ separately which together with its partial derivatives of order k and lower with respect to θ and ϕ is continuous everywhere, and the uniform modulus of continuity of the partial derivatives of order k does not exceed $Nk!(e^{e-1}-1)\,\Omega(\delta)$, where N is the larger of unity and $8M/\Omega(2^{3/2})$.*

LEMMA II. *If the kth-order partial derivatives of $g(x, y)$, $k \geqq 1$, above satisfy a Lipschitz condition with parameter Λ, those of $G(\theta, \phi)$ satisfy the inequalities*

$$\left| G^{i,k-i}(\theta_1, \phi_1) - G^{i,k-i}(\theta_2, \phi_2) \right| \leqq L\left(\left| \theta_1 - \theta_2 \right| + \left| \phi_1 - \phi_2 \right| \right),$$

$i = 0, 1, 2, \cdots, k$, everywhere, where $L = N'k!\,(e^{e-1}-1)$ and N' is the larger of Λ and M, the symbol $G^{i,k-i}$ being used to denote $\partial^k G/\partial\theta^i \partial\phi^{k-i}$.

These two lemmas are of sufficient importance to warrant somewhat detailed demonstrations. Since M is the maximum of $\left|g^{r,s}(x, y)\right|$ in the region $-1 \leqq x, y \leqq 1$ for $1 \leqq r+s \leqq k$, it follows from the law of the mean that

$$\left| g^{r,s}(x_1, y_1) - g^{r,s}(x_2, y_2) \right| \leqq M\left(\left| x_1 - x_2 \right| + \left| y_1 - y_2 \right| \right) \tag{14}$$

for $0 \leqq r+s \leqq k-1$ and all points of the region. Obviously, then, $g(x, y)$ and all its partial derivatives of order lower than k satisfy a Lipschitz condition. It is necessary in what follows to evaluate (so to speak) the condition (14) in terms of the uniform modulus $\Omega(\delta)$ of the partial derivatives of order exactly k. If $(x_1-x_2)^2+(y_1-y_2)^2 \leqq \delta^2$, then certainly $|x_1-x_2|+|y_1-y_2| \leqq \delta 2^{1/2}$. When $0 < \delta \leqq 2^{3/2}$, $2\,\Omega(\delta)/\delta \geqq \Omega(2^{3/2})/2^{3/2}$ by (13). Moreover, this same inequality holds without modification for $2^{3/2} \leqq \delta \leqq 2^{5/2}$ because $\Omega(\delta) = \Omega(2^{3/2})$ for these values of δ. (Obviously, the inequality can be adjusted for larger values of δ, but there is no need in this paper for values of δ greater than

$\pi 2^{1/2}$.) Consequently, when $0<\delta\leqq 2^{5/2}$, $0<\delta\leqq 2^{5/2}\,\Omega(\delta)/\,\Omega(2^{3/2})$ and $M(\,|x_1-x_2\,|$ $+\,|y_1-y_2\,|)\leqq 8M\,\Omega(\delta)/\,\Omega(2^{3/2})$. Therefore, $g(x,\,y)$ and all its partial derivatives of order k and lower have a uniform modulus of continuity nowhere exceeding $N\,\Omega(\delta)$, where N is the larger of unity and $8M/\,\Omega(2^{3/2})$.

Now form the partial derivatives of $G(\theta,\,\phi)$:

$$G^{1,0}(\theta,\,\phi)\;=\;g^{1,0}(x,\,y)(-\,\sin\theta),$$
$$G^{0,1}(\theta,\,\phi)\;=\;g^{0,1}(x,\,y)(-\,\sin\phi),$$
$$\cdots\cdots\cdots\cdots\cdots$$

$$(15)\;G^{i,k-i}(\theta,\,\phi)\;=\;\begin{cases}\displaystyle\sum_{s=1}^{k-i}\sum_{r=1}^{i}g^{r,s}(x,\,y)P_{r,i}Q_{s,i}/(r!s!),\;i=1,\cdots,k-1\geqq 1,\text{ or}\\[2ex]\displaystyle\sum_{s=1}^{k}g^{0,s}(x,\,y)Q_{s,0}/s!,\qquad\qquad i=0\;(k\geqq 1),\text{ or}\\[2ex]\displaystyle\sum_{r=1}^{i=k}g^{r,0}(x,\,y)P_{r,i}/r!,\qquad\qquad i=k\geqq 1,\end{cases}$$

where $P_{r,i}$ and $Q_{s,i}$ are polynomials of degree r in $\cos\theta$ and $\sin\theta$ and of degree s in $\cos\phi$ and $\sin\phi$, and both are independent of $g(x,\,y)$ and of each other. It has been shown by de la Vallée Poussin* that

$$|\,P_{r,i}\,|\;\leqq\;i!(e-1)^r,\;|\,Q_{s,i}\,|\;\leqq\;(k-i)!(e-1)^s.$$

Denote $|G^{i,k-i}(\theta_1,\,\phi_1)-G^{i,k-i}(\theta_2,\,\phi_2)\,|$ by D. Then if the first of the forms (15) be considered,

$$(16)\qquad D\;\leqq\;i!(k-i)!\sum_{s=1,r=1}^{k-i,\,i}\{\,|\,g^{r,s}(x_1,\,y_1)\;-\;g^{r,s}(x_2,\,y_2)\,|\,(e-1)^{r+s}/(r!s!)\}\,.$$

In this inequality the following facts are noted: $i!(k-i)!\leqq k!$; if $(\theta_1-\theta_2)^2$ $+(\phi_1-\phi_2)^2\leqq\delta^2$ then $(x_1-x_2)^2+(y_1-y_2)^2\leqq\delta^2$ also, and by the conclusion reached from (14) each of the absolute differences entering does not exceed $N\,\Omega(\delta)$; finally, $\sum(e-1)^{r+s}/(r!s!)\;\leqq\;\sum_1^\infty(e-1)^k/k!\;\leqq\;e^{e-1}-1$. Therefore, $D\leqq k!N(e^{e-1}-1)\,\Omega(\delta)$. The same is true in case either of the other forms in (15) is appropriate. Thus, the first lemma is proved.

If the kth-order partial derivatives of $g(x,\,y)$ satisfy a Lipschitz condition with parameter Λ then conditions (14) subsist for $r+s\leqq k$ with parameter N', where N' is the larger of Λ and M. Since $|x_1-x_2\,|+|y_1-y_2\,|\leqq|\theta_1-\theta_2\,|$ $+\,|\phi_1-\phi_2\,|$, it is apparent from (16) that the second lemma holds also.

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* de la Vallée Poussin, op. cit., pp. 67–68.

The preceding proofs can readily be adapted to demonstrations of the following lemmas:

LEMMA III. *Let $g(x, y)$ be a periodic function of period 2π in x alone which together with its partial derivatives of order $k \geqq 1$ and lower is continuous in the infinite strip $-\infty < x < \infty$, $-1 \leqq y \leqq 1$; let M be the maximum of the absolute values of these derivatives in the region; let the uniform modulus of continuity of the partial derivatives of order k be $\Omega(\delta)$. Then, if $\Omega(d) \neq 0$, where $d^2 = \pi^2 + 4$, $G(\theta, \phi) = g(\theta, \cos \phi)$ is a periodic function of period 2π in θ and in ϕ separately which with its partial derivatives of order k and lower is continuous everywhere, and the uniform modulus of continuity of the partial derivatives of order k nowhere exceeds $N''k!(e^{e-1}-1)\,\Omega(\delta)$, where N'' is the larger of unity and $2^{3/2}Md/\Omega(d)$.*

LEMMA IV. *If the kth-order partial derivatives of $g(x, y)$ in Lemma* III *satisfy a Lipschitz condition with parameter Λ, the kth-order partial derivatives of $G(\theta, \phi)$ satisfy relations*

$$\left| G^{i,k-i}(\theta_1, \phi_1) - G^{i,k-i}(\theta_2, \phi_2) \right| \leqq L\left(\left| \theta_1 - \theta_2 \right| + \left| \phi_1 - \phi_2 \right| \right),$$

where L is the constant of Lemma II.

The lemmas above will be used in §§5 and 6 to throw the burden of the proofs there on the theorems of §4.

4. **Degree of convergence of trigonometric approximation.** With the aid of the moduli of continuity discussed in the first part of the preceding article the function I_{pq} of §2 and functions analogous to it furnish trigonometric sums of type (1) approximating to a given periodic function. The following existence theorems exhibit the attainable degree of approximation by such sums to continuous functions and to functions having continuous partial derivatives.

THEOREM I.* *If $f(x, y)$ is a periodic function of period 2π in x and in y separately which everywhere satisfies a Lipschitz condition with parameter λ, then corresponding to every pair of positive integers m and n there exists a trigonometric sum $T_{mn}(x, y)$ of type* (1) *such that*

$$\left| f(x, y) - T_{mn}(x, y) \right| \leqq K\lambda(1/m + 1/n)$$

everywhere, where K is an absolute constant. (The conclusion is equally valid with K replaced by a suitable constant K_1 if $f(x, y)$ is such that

$$\left| f(x_1, y_1) - f(x_2, y_2) \right| \leqq \lambda \left| x_1 - x_2 \right| + \lambda \left| y_1 - y_2 \right|$$

everywhere.)

* The same theorem is given by Wilder, loc. cit.

The proof of this theorem is a straightforward extension of that for the corresponding theorem for one variable. In outline it is as follows: Given m and n, choose p and q as in §2 and construct the function $I_{pq}(x, y)$ for $f(x, y)$ by substituting $f(x, y)$ for $g(x, y)$ in (4), and take $T_{mn}(x, y) = I_{pq}(x, y)$; set up the difference (7); make use of the Lipschitz condition to replace the absolute difference in the integrand of (7) by $2\lambda(|u|+|v|) \geqq \lambda(4u^2+4v^2)^{1/2}$; split the resulting even integral into parts and apply (8) and (9) to each part to obtain $K\lambda(1/m+1/n)$ for an upper bound for the right-hand side of (7), where K is a combination of the c's.

THEOREM II. *If $f(x, y)$, periodic as in Theorem* I, *is merely continuous with modulus of continuity $\omega(\delta)$, then sums T_{mn} can be constructed so that*

$$|f(x, y) - T_{mn}(x, y)| \leqq K_2\omega(1/m + 1/n),$$

where K_2 is an absolute constant.

The proof of this theorem, also, will be sketched. Form $T_{mn}(x, y)$ as in the proof of Theorem I; replace the absolute difference in the integrand of (7) by $2\omega[(u^2+v^2)^{1/2}] \geqq \omega[(4u^2+4v^2)^{1/2}]$; split the resulting even integral up into

$$4h_{pq}\left\{\int_0^{1/q}\int_0^{1/p} + \int_{1/q}^{\pi/2}\int_{1/p}^{\pi/2} + \int_{1/q}^{\pi/2}\int_0^{1/p} + \int_0^{1/q}\int_{1/p}^{\pi/2}\right\},$$

and apply the properties (10), (11), (12), and (13) of $\omega(\delta)$ and the inequalities (8) and (9) to h_{pq} and to the integrals separately, noting particularly that $\omega[(u^2+v^2)^{1/2}] \leqq \omega(u+v) \leqq \omega(u)+\omega(v)$, and that for $u \geqq 1/p$, $2\omega(1/p)/(1/p) \geqq \omega(u)/u$, so that $\omega(u) \leqq 2pu\omega(1/p)$, a similar inequality holding for $\omega(v)$ when $v \geqq 1/q$.

THEOREM III. *Let $f(x, y)$ be a periodic function of period 2π in x and in y separately for which the partial derivatives $f^{i,k-i}(x, y)$, $i = 0, 1, \cdots, k$, all exist and are everywhere continuous, and let p and q be two such integers that $2p-2 \leqq m \leqq 2p$ and $2q-2 \leqq n \leqq 2q$, where m and n are two given positive integers. If the partial derivatives of order k are such that the $k+1$ functions*

$$I_{i,k-i}(x, y) = h_{pq}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f^{i,k-i}(x + 2u, y + 2v)F_{pq}(u, v)dudv$$

satisfy the inequalities

$$|f^{i,k-i}(x, y) - I_{i,k-i}(x, y)| \leqq \epsilon$$

everywhere, where ϵ is some finite positive constant or zero, then there exists a sum $T(x, y)$ of type (1) *of order at most m in x and n in y such that*

$$| f(x, y) - T(x, y) | \leqq K_1^k \epsilon (1/m + 1/n)^k$$

everywhere.

Let

$$t_1(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(x + 2u, y + 2v) F_{pq}(u, v) du dv.$$

Since conditions for differentiating under the integral sign are fulfilled it is immediately apparent that

$$t_1^{i,k-i}(x, y) = I_{i,k-i}(x, y) \qquad (i = 0, 1, \cdots, k).$$

Furthermore, $t_1(x, y)$ is a sum of type (1) of order not exceeding m in x and n in y. Form the function

$$R_1(x, y) = f(x, y) - t_1(x, y).$$

By hypothesis the partial derivatives of order k of R_1 satisfy the relations $|R_1^{i,k-i}(x, y)| \leqq \epsilon$ everywhere, whence by the law of the mean

$$| R_1^{i,j-i}(x_1, y_1) - R_1^{i,j-i}(x_2, y_2) | \leqq \epsilon(| x_1 - x_2 | + | y_1 - y_2 |),$$

$i = 0, 1, \cdots, j = k - 1$. Theorem I is now applicable to each of the j functions $R_1^{i,j-i}(x, y)$ with λ replaced by ϵ and the $T_{mn}(x, y)$ replaced by the functions

$$I_{i,j-i}(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} R_1^{i,j-i}(x + 2u, y + 2v) F_{pq}(u, v) du dv$$

corresponding to $R_1^{i,j-i}(x, y)$. The theorem yields the inequalities

$$| R_1^{i,j-1}(x, y) - I_{i,j-i}(x, y) | \leqq K_1 \epsilon (1/m + 1/n).$$

Now let

$$t_2(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} R_1(x + 2u, y + 2v) F_{pq}(u, v) du dv$$

and repeat the process begun on $t_1(x, y)$ and $R_1(x, y)$, and thereby construct a sequence of sums $t_1(x, y), t_2(x, y), \cdots, t_k(x, y)$, all of type (1) and of order at most m in x and n in y, and a sequence of functions $R_1(x, y), R_2(x, y), \cdots, R_k(x, y)$ in which $R_k(x, y) = f(x, y) - t_1(x, y) - t_2(x, y) - \cdots - t_k(x, y)$ satisfies the inequality

$$| R_k(x_1, y_1) - R_k(x_2, y_2) | \leqq K_1^{k-1} \epsilon (1/m + 1/n)^{k-1} (| x_1 - x_2 | + | y_1 - y_2 | .$$

Consequently, by a final application of Theorem I, there exists a sum $t_{k+1}(x, y)$ of type (1) and of order at most m in x and n in y such that

$$| R_k(x, y) - t_{k+1}(x, y) | \leqq K_1^k \epsilon (1/m + 1/n)^k$$

everywhere. In this inequality let $t_1(x, y)+t_2(x, y)+\cdots+t_{k+1}(x, y)$ be denoted by $T(x, y)$, a trigonometric sum of type (1) of order at most m in x and n in y, and the theorem is proved.

If, now, the kth-order partial derivatives of $f(x, y)$ are continuous with uniform modulus of continuity $\Omega(\delta)$, then by Theorem II each of these derivatives can be approximated by a sum of type (1) given precisely by the $I_{i,k-i}(x, y)$ of Theorem III, and ϵ will have the value $K_2\Omega(1/m+1/n)$; if these derivatives satisfy a Lipschitz condition with parameter Λ, ϵ will have the form $K\Lambda(1/m+1/n)$. Consequently the following theorem is true:

THEOREM IV. *If $f(x, y)$, periodic of period 2π in x and in y separately, is such that its kth-order partial derivatives are everywhere continuous with uniform modulus of continuity $\Omega(\delta)$, then corresponding to every pair of positive integers m and n there exists a trigonometric sum $T_{mn}(x, y)$ of type (1) such that*

$$| f(x, y) - T_{mn}(x, y) | \leqq K_2 K_1^k (1/m + 1/n)^k \Omega(1/m + 1/n)$$

everywhere, and if these derivatives satisfy a Lipschitz condition with parameter Λ the right-hand side of the inequality becomes $K_1^k K\Lambda(1/m+1/n)^{k+1}$.

5. **Degree of convergence of polynomial approximation.** Polynomial approximations to a function $f(x, y)$ are effected by obtaining trigonometric approximations to the transformed function

$$F(\theta, \phi) = f(\cos\theta, \cos\phi)$$

from the theorems of the preceding article, using the relations of Lemmas I and II and the conclusion of the following lemma.*

LEMMA V. *If $F(\theta, \phi)$ is an even function of θ and ϕ separately and if there exists a sum $T(\theta, \phi)$ of type (1) such that*

$$| F(\theta, \phi) - T(\theta, \phi) | \leqq \epsilon$$

everywhere, then there exists a sum $t(\theta, \phi)$ of the same type, of order not higher than that of $T(\theta, \phi)$, and devoid of sines of multiples of either θ or ϕ, such that

$$| F(\theta, \phi) - t(\theta, \phi) | \leqq \epsilon$$

everywhere.

On account of the evenness of $F(\theta, \phi)$ and the inequality in the hypothesis of the lemma,

* Cf. Jackson, *Colloquium*, p. 13.

$$\begin{aligned}&\left| F(\theta, \phi) - \tfrac{1}{4}\{T(\theta, \phi) + T(\theta, -\phi) + T(-\theta, \phi) + T(-\theta, -\phi)\}\right| \\ &\quad \leqq \tfrac{1}{4}\{\left|F(\theta, \phi) - T(\theta, \phi)\right| + \left|F(\theta, -\phi) - T(\theta, -\phi)\right| \\ &\quad\quad + \left|F(-\theta, \phi) - T(-\theta, \phi)\right| + \left|F(-\theta, -\phi) - T(-\theta, -\phi)\right|\} \\ &\quad \leqq \tfrac{1}{4}4\epsilon = \epsilon.\end{aligned}$$

But the sum in braces on the left is $4T(\theta, \phi)$ with all the terms containing sines removed; hence the lemma is true.

Proofs of theorems paralleling those of §4 all follow the same scheme. Of these theorems the one based on Theorem IV is the most general; it is sufficiently typical to warrant omitting the others.

THEOREM V. *Let $f(x, y)$ together with its partial derivatives of order $k \geqq 1$ and lower be continuous in the square region $-1 \leqq x, y \leqq 1$; let M be the maximum of the absolute values of these derivatives in the region; let $\Omega(\delta) \not\equiv 0$ be the uniform modulus of continuity of the derivatives of order k. Then corresponding to every pair of positive integers m and n there exists a polynomial $P_{mn}(x, y)$ of type (2) such that*

$$\left| f(x, y) - P_{mn}(x, y)\right| \leqq K_3 N(1/m + 1/n)^k \Omega(1/m + 1/n)$$

*throughout the region, where $K_3 = K_2 K_1^k k!(e^{e-1} - 1)$ and N is the larger of unity and $8M/\Omega(2^{3/2})$.**

Under the hypotheses of the theorem and on account of Lemma I, $F(\theta, \phi) = f(\cos\theta, \cos\phi)$ is a periodic function of period 2π in θ and in ϕ separately having kth-order partial derivatives everywhere continuous with a uniform modulus of continuity which does not exceed $Nk!(e^{e-1} - 1)\,\Omega(\delta)$. By Theorem IV there exists a sum of type (1) which everywhere approximates $F(\theta, \phi)$ within an error nowhere exceeding that assigned for $P_{mn}(x, y)$ in Theorem V. Since $F(\theta, \phi)$ is even in θ and in ϕ separately, there exists by Lemma V a sum $T_{mn}(\theta, \phi)$ of the same type containing no sines of either θ or ϕ and giving at least as good an approximation; this $T_{mn}(\theta, \phi)$ is a polynomial in $\cos\theta$ and $\cos\phi$ of degree not exceeding m in $\cos\theta$ and n in $\cos\phi$: $T_{mn}(\theta, \phi) = P_{mn}(\cos\theta, \cos\phi)$. Consequently, $f(x, y)$ is approximated by $P_{mn}(x, y)$ within an error not exceeding that permitted in the theorem. If $\Omega(\delta) \equiv 0$ then $f(x, y)$ is itself a polynomial of degree at most k, in which case the above theorem does not, and need not, apply.

6. **Degree of convergence of mixed approximation.** In case $f(x, y)$ is periodic in one variable only and satisfies conditions of continuity in an infinite strip of finite width, methods analogous to those of §5 lead to approximations

* For a theorem on polynomial approximation related to this one, but neither containing it nor contained in it, see P. Montel, *Sur les polynomes d'approximation*, Bulletin de la Société Mathématique de France, vol. 46 (1918), pp. 151–192; p. 191.

in the form of sums which are trigonometric in that variable and polynomial in the other: mixed sums of type (3). As in §5 one representative theorem will suffice.

THEOREM VI. *Let $f(x, y)$ be a periodic function of period 2π in x alone which together with its partial derivatives of order $k \geqq 1$ and lower is continuous in the region $-\infty < x < \infty$, $-1 \leqq y \leqq 1$; let M be the maximum of the absolute values of the partial derivatives in the region; let $\Omega(\delta)$ be the uniform modulus of continuity of the partial derivatives of order k; let $\Omega(d) \neq 0$, where $d^2 = \pi^2 + 4$. Then corresponding to every pair of positive integers m and n there exists a mixed sum $H_{mn}(x, y)$ of type (3) such that*

$$| f(x, y) - H_{mn}(x, y) | \leqq K_3 N''(1/m + 1/n)^k \Omega(1/m + 1/n)$$

throughout the region, where K_3 is the same as in Theorem V *and N'' is the larger of unity and $2^{3/2}Md/\Omega(d)$.*

Let $F(\theta, \phi) = f(\theta, \cos\phi)$. By Lemma III, $F(\theta, \phi)$ is a periodic function of period 2π in θ and in ϕ separately, which has everywhere continuous partial derivatives of order $k \geqq 1$ whose uniform modulus of continuity does not exceed $N''k!(e^{e-1}-1)\,\Omega(\delta)$. Hence by Theorem IV there exists a sum of type (1) which approximates $F(\theta, \phi)$ within the error assigned in the above theorem. Since $F(\theta, \phi)$ is an even function in ϕ, this can be replaced by a sum $T_{mn}(\theta, \phi)$ containing no sines of ϕ and, consequently, is expressible as a polynomial of degree not exceeding n in $\cos\phi$: a mixed sum of type (3), $T_{mn}(\theta, \phi) = H_{mn}(\theta, \cos\phi)$. The proof is an appropriate simplification of that of Lemma V. Therefore, $f(x, y)$ is approximated by $H_{mn}(x, y)$ within the error given. If $\Omega(d) = 0$, $f(x, y)$ does not contain x and is a polynomial of degree at most k in y. As in §5 the above theorem does not, and need not, apply in this case.

Upon examination it will be noticed that, except for the absolute constants involved, the inclusive theorems on the approximation to a function having continuous partial derivatives of order k (Theorems IV, V, and VI) reduce to the simpler theorems for an ordinary continuous function by placing k equal to zero.

II. Approximation by sums of surface spherical harmonics

7. Degree of convergence of Laplace's series. Let θ and ϕ be the co-latitude and longitude, respectively, of a point on the surface of a sphere of unit radius, and let $F(\theta, \phi)$ be a real, single-valued, integrable point function on the sphere. The partial sum of degree n of the expansion of $F(\theta, \phi)$ in Laplace's series* is

* See, e.g., Byerly, *Fourier's Series and Spherical Harmonics*, Boston, Ginn and Co., 1895, p. 211.

$$(17) \qquad S_n(\theta, \phi) = \sum_{i=0}^{n} \frac{2i+1}{4\pi} \int_0^{2\pi} \int_0^{\pi} F(\theta', \phi') P_i(\cos\gamma)(\sin\theta')d\theta' d\phi',$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$ and P_i is Legendre's polynomial (the Legendrian) of degree i.

By means of an existence theorem (cf. §8 below) Gronwall* gave an elegant proof concerning the degree of convergence of $S_n(\theta, \phi)$ to $F(\theta, \phi)$. His method, however, does not seem to permit extension to functions having continuous derivatives beyond those of the first order. The attack below through the medium of the mean-value function used by Dirichlet†, Darboux‡, and others, leads readily to an extended theorem. The success of the attack is due largely to the two accompanying lemmas.

LEMMA VI. *Let* $\sigma_n(x) = (1/2)\sum_0^n (2i+1) P_i(x)$. *If* $g(x)$ *is an integrable function such that* $|g(x)| \leqq G$ *throughout the region* $-1 \leqq x \leqq 1$, *then*

$$\left| \int_{-1}^{1} g(x)\sigma_n(x)dx \right| \leqq c'Gn^{1/2}, \quad n \geqq 1,$$

where c' *is an absolute constant.*

LEMMA VII. *If* $p_n(x)$ *is a polynomial of degree at most* n *in* x *then*

$$\int_{-1}^{1} p_n(x)\sigma_n(x)dx = p_n(1)$$

for all positive integral values of n.

The first of these lemmas is an adaptation of the fact that

$$\lim_{n\to\infty} \left\{ (1/n)^{1/2} \int_{-1}^{1} |\sigma_n(x)| dx \right\} = 2(2/\pi)^{1/2},$$

a fact proved by Gronwall.§ The second follows on substituting the identity¶

$$\sigma_n(x) = \tfrac{1}{2}(d/dx)[P_{n+1}(x) + P_n(x)]$$

in the integrand and integrating by parts. Thus

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* T. H. Gronwall, *On the degree of convergence of Laplace's series*, these Transactions, vol. 15 (1914), pp. 1–30; see pp. 14–23.

† Dirichlet, *Sur les séries dont le terme général dépend de deux angles* · · · , Journal für Mathematik, vol. 17 (1837), pp. 35–56.

‡ Darboux, same title as the preceding, Journal de Mathématiques, (2), vol. 19 (1874), pp. 1–18.

§ Gronwall, loc. cit., pp. 3–14; also *Über die Laplacesche Reihe*, Mathematische Annalen, vol. 74 (1913), pp. 213–270; see pp. 222–230.

¶ See, for example, Byerly, op. cit., p. 180.

$$\int_{-1}^{1} p_n(x)\sigma_n(x)dx = \tfrac{1}{2}\left[p_n(x)\left\{P_{n+1}(x) + P_n(x)\right\}\right]_{-1}^{1} - \tfrac{1}{2}\int_{-1}^{1} p_n'(x)\left[P_{n+1}(x) + P_n(x)\right]dx.$$

The value of the first term on the right is $p_n(1)$. Since $p'_n(x)$ is of lower degree than either $P_{n+1}(x)$ or $P_n(x)$, it is orthogonal to each and, consequently, to their sum; hence the second term on the right is zero, and the lemma is proved.

Let the system of coördinates be rotated to place the pole at the point (θ, ϕ); let the principal meridian be any fixed great circle through this pole; let x and y be the new geographic coördinates. Then the pole (θ, ϕ) transforms into the point $(0, y)$; $\cos\gamma$ into $\cos x$; $S_n(\theta, \phi)$ into a constant, say $s_n(0)$; $F(\theta', \phi')$ into a new function $f(x, y)$; $F(\theta, \phi)$ into $f(0, y)$. Consequently, (17) becomes

$$\begin{aligned} S_n(\theta, \phi) = s_n(0) &= \sum_0^n \left\{ \frac{2i+1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(x, y) P_i(\cos x)(\sin x)dxdy \right\} \\ &= \int_0^{\pi} \left[\sum_0^n \frac{2i+1}{2} P_i(\cos x) \right] \left[\frac{1}{2\pi} \int_0^{2\pi} f(x, y)dy \right] (\sin x)dx \\ &= \int_0^{\pi} \Phi(x; \theta, \phi)\sigma_n(\cos x)(\sin x)dx, \end{aligned}$$

where $\Phi(x; \theta, \phi) = (1/(2\pi))\int_0^{2\pi} f(x, y)dy$ is the *mean value* of F on a circle of curved radius (polar distance) x with center (pole) at the point (θ, ϕ).

With the aid of the last two lemmas the following theorem is established.

THEOREM VII. *Let $F(\theta, \phi)$ be a real, single-valued, integrable point function on the unit sphere, and let $\Phi(x; \theta, \phi)$ be the mean value of F on a circle of curved radius x with center at (θ, ϕ). If $F(\theta, \phi)$ is of such a nature that corresponding to a positive constant ϵ_n there exists a polynomial $p_n(\cos x)$ of degree at most n in $\cos x$ satisfying the inequality*

$$\left| \Phi(x; \theta, \phi) - p_n(\cos x) \right| \leqq \epsilon_n$$

for all values of x, then

$$\left| S_n(\theta, \phi) - \Phi(0; \theta, \phi) \right| \leqq c\epsilon_n n^{1/2}$$

for all positive integral values of n, where c is an absolute constant.

Let $\Phi(x; \theta, \phi) - p_n(\cos x)$ be denoted by $g(x)$. By hypothesis $|g(x)| \leqq \epsilon_n$; therefore, by Lemma VI,

$$\left| \int_0^\pi g(x)\sigma_n(\cos x)(\sin x)dx \right| \leqq \epsilon_n \int_{-1}^{1} |\sigma_n(x)|\, dx \leqq c'\epsilon_n n^{1/2}.$$

In other words,

$$\left| \int_0^\pi \Phi(x;\theta,\phi)\sigma_n(\cos x)(\sin x)dx - \int_0^\pi p_n(\cos x)\sigma_n(\cos x)(\sin x)dx \right| \leqq c'\epsilon_n n^{1/2}.$$

Here the first integral is $s_n(0)$ and the second, by virtue of Lemma VII, is $p_n(\cos 0)$. The inequality, then, takes the form

$$|\, s_n(0) - p_n(\cos 0)\,| \leqq c'\epsilon_n n^{1/2}.$$

It was assumed in the hypothesis, however, that

$$|\, \Phi(0;\theta,\phi) - p_n(\cos 0)\,| \leqq \epsilon_n.$$

By combining these last two inequalities the following inequality is obtained:

$$|\, \Phi(0;\theta,\phi) - s_n(0)\,| \leqq \epsilon_n(1 + c'n^{1/2}) \leqq c\epsilon_n n^{1/2}.$$

But $s_n(0) = S_n(\theta,\phi)$, and the theorem is proved.

Suppose now that $F(\theta,\phi)$ is continuous on the surface of the sphere with modulus of continuity $\omega(\delta)$; i.e.,

$$|\, F(\theta_1,\phi_1) - F(\theta_2,\phi_2)\,| \leqq \omega(\delta)$$

for all points for which $\Gamma \leqq \delta$, where Γ is the shorter great-circle distance between the points. If $\omega(\delta)$ does not exceed $\lambda\delta$, where λ is a positive constant, $F(\theta,\phi)$ will be said to satisfy a Lipschitz condition with parameter λ. Since each point (x, y) on the sphere can be thought of as having infinitely many alternative pairs of coördinates, $(x+2\mu\pi,\ y+2\nu\pi)$, $(-x+2\mu\pi,\ y+(2\nu+1)\pi)$, where μ and ν are arbitrary integers, positive, negative, or zero, $f(x, y)$, considered as a point function on the sphere, is periodic of period 2π in x and y separately and $\Phi(x;\theta,\phi)$ is a periodic even function of x of period 2π having the same modulus of continuity, $\omega(\delta)$. For a fixed pole, then, it can be inferred from a well known theorem* and from the analogue of Lemma V for functions of a single variable that corresponding to every positive integer n there exists a trigonometric sum $T_n(x)$ *containing only cosine terms*, of order at most n in x, such that

$$|\, \Phi(x;\theta,\phi) - T_n(x)\,| \leqq K'\omega(2\pi/n) \tag{18}$$

* See, e.g., Jackson, *Colloquium*, p. 7; *On the approximate representation of an indefinite integral* · · · , these Transactions, vol. 14 (1913), pp. 343–364; p. 350.

where K' is an absolute constant. (If $F(\theta, \phi)$ satisfies a Lipschitz condition the absolute error in (18) does not exceed $K''\lambda/n$, where K'' is an absolute constant.) Since $T_n(x)$ contains only cosine terms it is a polynomial $p_n(\cos x)$ of degree at most n in $\cos x$.

The selection of $T_n(x) = p_n(\cos x)$ depends on the choice of the pole, but no matter what point is chosen for pole the accompanying polynomial in $\cos x$ satisfies (18). The hypotheses of Theorem VII are fulfilled at every point on the sphere. On account of the continuity of $F(\theta, \phi)$, evidently $\Phi(0; \theta, \phi) = F(\theta, \phi)$. Hence the theorem stated below is true.

THEOREM VIII. *If $F(\theta, \phi)$ is continuous with modulus of continuity $\omega(\delta)$ on the surface of the sphere, then*

$$| F(\theta, \phi) - S_n(\theta, \phi) | \leqq cK'\omega(2\pi/n)n^{1/2}, \ n > 0,$$

for all points on the sphere, where c and K' are absolute constants. If $F(\theta, \phi)$ satisfies a Lipschitz condition with parameter λ, the absolute error does not exceed $cK''\lambda/n^{1/2}$, where K'' is an absolute constant.

COROLLARY I. *If $F(\theta, \phi)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta \to 0} \omega(\delta)/\delta^{1/2} = 0$, Laplace's series converges uniformly to $F(\theta, \phi)$ over the surface of the sphere.*

The theorem above, which is substantially the same as that of Gronwall, permits the following extension:

THEOREM IX. *Let $F(\theta, \phi)$ be continuous and such that the kth-order derivatives $(\partial^k/\partial s^k)F(\theta, \phi)$ with respect to arc-length exist on every great circle on the sphere with moduli of continuity not exceeding a common upper bound $\omega(\delta)$ such that $\lim_{\delta \to 0} \omega(\delta) = 0$. Then the partial sum of order n, $S_n(\theta, \phi)$, of Laplace's series for $F(\theta, \phi)$, satisfies the inequality*

$$| F(\theta, \phi) - S_n(\theta, \phi) | \leqq A\omega(2\pi/n)(1/n)^k n^{1/2}, \ n > 0,$$

where A is an absolute constant.

Under the hypotheses it follows on differentiating under the integral sign that $(\partial^k/\partial x^k)\Phi(x; \theta, \phi)$ is a continuous function of x with modulus of continuity not exceeding $\omega(\delta)$. Under these circumstances $\Phi(x; \theta, \phi)$ satisfies the conditions of a theorem* which in substance states that there exists a polynomial $p_n(\cos x)$ which approximates Φ with an absolute error not exceeding $A'\omega(2\pi/n)\ (1/n)^k$, where A' is an absolute constant. Consequently, by the argument used in the proof of Theorem VIII, the present theorem holds.

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* Jackson, *Colloquium*, p. 12.

8. **An existence theorem.** The partial sum of Laplace's series is a special form of the general sum of degree n of surface spherical harmonics*:

$$(19) \qquad Y_n(\theta, \phi) = \sum_{i=0}^{n} \sum_{j=0}^{i} \{ [A_{ij} \cos j\phi + B_{ij} \sin j\phi] P_i^j(\cos\theta) \};$$

the A's and B's are real constants and

$$P_i^j(\cos\theta) = (\sin j\theta)\ (d/d\cos\theta)^j P_i(\cos\theta)$$

is the associated function of order j and degree i.

Gronwall† has shown that there exists a sum of the form (19) which under certain conditions approximates a given function more closely than does the partial sum of Laplace's series for the function. (Cf. (4), (5), and (6) of §2.) Let

$$(20) \qquad T_n(\theta, \phi) = h_p \int_0^{2\pi} \int_0^{\pi} F(\theta', \phi') g_p(\gamma) (\sin\theta') d\theta' d\phi',$$

where γ is the great-circle distance between (θ', ϕ') and (θ, ϕ),

$$g_p(\gamma) = [(\sin(p\gamma/2))/(\sin(\gamma/2))]^4,$$

and

$$1/h_p = \int_0^{2\pi} \int_0^{\pi} g_p(\gamma)(\sin\theta') d\theta' d\phi'.$$

That $T_n(\theta, \phi)$, thus defined by Gronwall, is of the form (19) is established from the following facts: $[(\sin(p\gamma/2))/(\sin(\gamma/2))]^2 = (1-\cos p\gamma)/(1-\cos\gamma)$ is a cosine sum of order $p-1$, so that $g_p(\gamma)$ is such a sum of order $2p-2$; g_p is therefore a polynomial of degree $2p-2$ in $\cos\gamma$ and, consequently, is expressible as a linear combination of Legendrians in $\cos\gamma$, $\sum_0^{2p-2} a_i P_i(\cos\gamma)$; since a Legendrian, $P_i(\cos\gamma)$, is a surface spherical harmonic of degree i, it follows that $T_n(\theta, \phi)$ is a surface spherical harmonic sum of degree not exceeding n when p is an integer such that $2p-2 \leqq n \leqq 2p$. Gronwall proved that if $F(\theta, \phi)$ has a modulus of continuity $\omega(\delta)$ on the sphere, $T_n(\theta, \phi)$ satisfies the inequality

$$|F(\theta, \phi) - T_n(\theta, \phi)| \leqq B'\omega(1/n)$$

for all points on the sphere, where B' is an absolute constant. This theorem can be extended to include a function having continuous directional derivatives.

* Byerly, op. cit., p. 197.

† Gronwall, these Transactions, loc. cit., pp. 14–23.

THEOREM X. *If $F(\theta, \phi)$ has at every point of the sphere continuous first-order directional derivatives as described in Theorem* IX *with moduli of continuity not exceeding $\omega(\delta)$, where $\lim_{\delta \to 0}\omega(\delta) = 0$, then corresponding to every positive integer n there exists a sum $T_n(\theta, \phi)$ of form* (19), *of degree at most n, such that*

$$| F(\theta, \phi) - T_n(\theta, \phi) | \leqq B\omega(1/n)(1/n)$$

for all points on the sphere, where B is an absolute constant.

[Let (θ, ϕ) be chosen for a new pole of coördinates; let $\Phi(x; \theta, \phi)$ be the mean value of F on a circle of curved radius x with center at (θ, ϕ). Let the definition of $T_n(\theta, \phi)$ given by (20) be subjected to the following modifications: let p be an integer such that $3p-3 \leqq n \leqq 3p$, let

$$g_p(\gamma) = [(\sin\ (p\gamma/2))/(\sin\ (\gamma/2))\,]^6,$$

and let a factor 2π be divided out of the corresponding integral defining $1/h_p$. From the remarks accompanying the definition (20) it is apparent that $T_n(\theta, \phi)$ thus modified is also a surface spherical harmonic sum of degree not exceeding n. At the pole (θ, ϕ), then,

$$T_n(\theta, \phi) = h_p \int_0^\pi \Phi(x; \theta, \phi) g_p(x)(\sin x)dx,$$

with

$$1/h_p = \int_0^\pi g_p(x)(\sin x)dx.$$

From this last equation evidently

$$\Phi(0; \theta, \phi) = h_p \int_0^\pi \Phi(0; \theta, \phi) g_p(x)(\sin x)dx.$$

Consequently

$$T_n(\theta, \phi) - \Phi(0; \theta, \phi) = h_p \int_0^\pi [\Phi(x; \theta, \phi) - \Phi(0; \theta, \phi)]g_p(x)(\sin x)dx;$$

whence by the law of the mean, since $\Phi_x(x; \theta, \phi)$ is continuous in x,

$$T_n(\theta, \phi) - \Phi(0; \theta, \phi) = h_p \int_0^\pi x\Phi_x(qx; \theta, \phi) g_p(x)(\sin x)dx,$$

where $0 < q < 1$. Inasmuch as $\Phi(x; \theta, \phi)$ is an even function possessing a continuous derivative with respect to x it follows that $\Phi_x(0; \theta, \phi) = 0$. Therefore

$$\left| \Phi_x(qx; \theta, \phi) \right| = \left| \Phi_x(qx; \theta, \phi) - \Phi_x(0; \theta, \phi) \right|$$
$$\leqq \omega(qx) \leqq \omega(x).$$

Since, also, $\Phi(0; \theta, \phi) = F(\theta, \phi)$,

$$\left| T_n(\theta, \phi) - F(\theta, \phi) \right| \leqq h_p \int_0^\pi x\omega(x) g_p(x)(\sin x)dx.$$

By a method analogous to that suggested in the outline of the proof of Theorem II in §4, the right-hand side of this inequality does not exceed $B \cdot \omega(1/n)\,(1/n)$. Since the hypotheses are assumed to hold at every point (θ, ϕ) of the sphere so also does the conclusion.

9. **A problem of closest approximation in terms of surface spherical harmonics.** As in other cases of approximation by means of orthogonal functions, it is easily demonstrated that the particular choice of the coefficients A_{ij} and B_{ij} in the general surface spherical harmonic sum $Y_n(\theta, \phi)$ which minimizes the integral

$$\int_0^{2\pi} \int_0^\pi [F(\theta, \phi) - Y_n(\theta, \phi)]^2 (\sin\theta) d\theta d\phi,$$

where $F(\theta, \phi)$ is a given continuous function, is that choice which yields the partial sum $S_n(\theta, \phi)$ of Laplace's series for $F(\theta, \phi)$; there is one and only one choice of the coefficients which produces a minimum value of the integral.

As in the case of polynomial and of ordinary trigonometric approximation the above problem can be generalized into a problem of minimizing the integral of a power other than the square of the absolute discrepancy. Jackson* has given a general existence theorem which shows in particular that if $p_1(x), p_2(x), \cdots, p_k(x)$ is any set of k linearly independent continuous functions of x in an interval $a \leqq x \leqq b$ and $f(x)$ is continuous in this interval, then there exists one and only one choice of the coefficients in the linear combination $\sum_1^k c_i p_i(x)$ which minimizes the integral

$$\int_a^b \left| f(x) - \sum_1^k c_i p_i(x) \right|^m dx,$$

where m is any number greater than unity. By suitable adaptation the same method yields a proof of

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* Jackson, *A generalized problem in weighted approximation*, these Transactions, vol. 26 (1924), pp. 133–154; see pp. 133–138.

THEOREM XI. *Let $F(\theta, \phi)$ be a continuous single-valued function on the unit sphere, and let $Y_n(\theta, \phi)$ be a general surface spherical harmonic sum of degree n. There exists one and only one choice of the coefficients A_{ij} and B_{ij} which will render the integral*

$$\int_0^{2\pi} \int_0^{\pi} |F(\theta, \phi) - Y_n(\theta, \phi)|^m (\sin\theta) d\theta d\phi$$

a minimum when $m > 1$. Such a sum $Y_n(\theta, \phi)$ is called the approximating sum for $F(\theta, \phi)$ corresponding to exponent m.

The proof of existence, apart from the question of uniqueness, holds also for $0 < m \leqq 1$.

10. **Convergence of the approximating sum.** To begin with, it is to be observed that $Y_n(\theta, \phi)$ is a trigonometric sum of order n of type (1), §1. Hence Bernstein's theorem is applicable: if $|Y_n(\theta, \phi)| \leqq L$ over the entire sphere, then also $|(\partial/\partial\theta) Y_n(\theta, \phi)| \leqq nL$ and $|(\partial/\partial\phi) Y_n(\theta, \phi)| \leqq nL$ over the entire sphere. Since by an arbitrary rotation of coördinates which places the pole at the point (θ, ϕ), Y_n is transformed into another sum of the same character, the statement $|(\partial/\partial\theta) Y_n| \leqq nL$ can be given the more general form $|(\partial/\partial s) Y_n| \leqq nL$, where s is along any great circle through the point (θ, ϕ). With this observation a device used in connection with other problems* becomes available for finding conditions on the function $F(\theta, \phi)$ sufficient to insure uniform convergence of its approximating sum.

Let $F(\theta, \phi)$ be continuous on the sphere; for fixed n let $Y_n(\theta, \phi)$ be its approximating sum corresponding to exponent m; let γ_n be the minimum attained by the integral

$$\int_0^{2\pi} \int_0^{\pi} |F(\theta, \phi) - Y_n(\theta, \phi)|^m (\sin\theta) d\theta d\phi. \tag{21}$$

Suppose that there exists another sum $y_n(\theta, \phi)$ of the same type (19) such that

$$|F(\theta, \phi) - y_n(\theta, \phi)| \leqq \epsilon_n$$

everywhere on the sphere, where ϵ_n depends only on n. Place

$$\rho_n(\theta, \phi) = Y_n(\theta, \phi) - y_n(\theta, \phi)$$

and

$$r_n(\theta, \phi) = F(\theta, \phi) - y_n(\theta, \phi).$$

* See, e.g., Jackson, *On the convergence of certain trigonometric and polynomial approximations*, these Transactions, vol. 22 (1921), pp. 158–166; *Colloquium*, Chapter III. For an application of the method to a problem of approximation in two variables, see E. Carlson, *On the convergence of trigonometric approximations for a function of two variables*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 639–641.

Now, ρ_n is continuous; let μ_n be the maximum of its absolute value and (θ_0, ϕ_0) a point at which it is attained: $\mu_n = |\rho_n(\theta_0, \phi_0)|$. Let the point (θ, ϕ) be restricted to a circle R of curved radius $1/(2n)$ with center at (θ_0, ϕ_0). If (θ, ϕ) is distinct from (θ_0, ϕ_0), ρ_n is a continuous function of s with continuous derivatives with respect to s along the great circle joining the points. By the law of the mean,

$$|\rho_n(\theta, \phi) - \rho_n(\theta_0, \phi_0)| = |\partial\rho_n'/\partial s| s$$

where ρ_n' is a value of ρ_n at a point on the arc between the given points. Consequently, whether (θ, ϕ) is distinct from (θ_0, ϕ_0) or not,

$$|\rho_n(\theta, \phi) - \rho_n(\theta_0, \phi_0)| \leqq n\mu_n/(2n) = \mu_n/2.$$

Therefore, $|\rho_n(\theta, \phi)| \geqq \mu_n/2$ for all points in R.

For the moment let it be supposed that $\epsilon_n \leqq \mu_n/4$, the contrary case being considered presently. Then, since $|r_n(\theta, \phi)| \leqq \epsilon_n \leqq \mu_n/4$ everywhere and $|\rho_n(\theta, \phi)| \geqq \mu_n/2$ in R, it follows that $|r_n - \rho_n| \geqq \mu_n/4$ in R, and, consequently, that

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} |F(\theta, \phi) - Y_n(\theta, \phi)|^m (\sin\theta) d\theta d\phi \\
&\quad = \int_0^{2\pi}\int_0^{\pi} |r_n - \rho_n|^m (\sin\theta) d\theta d\phi \\
&\quad \geqq (\mu_n/4)^m \int_R (\sin\theta) d\theta d\phi = (\mu_n/4)^m 2\pi \int_0^{1/(2n)} (\sin\theta) d\theta \\
&\quad = (\mu_n/4)^m 4\pi \sin^2 [1/(4n)].
\end{aligned}$$

Therefore, $\gamma_n \geqq 4\pi(\mu_n/4)^m \sin^2[1/(4n)]$. But, since $Y_n(\theta, \phi)$ minimizes the integral (21), $\gamma_n \leqq 4\pi\epsilon_n^m$, and it follows that

$$\mu_n \leqq 4\{\gamma_n/(4\pi \sin^2 [1/(4n)])\}^{1/m} \leqq 4\{\sin^2 [1/(4n)]\}^{-1/m}\epsilon_n.$$

In the contrary case $\epsilon_n > \mu_n/4$, certainly $\mu_n < 4\epsilon_n$, so that in either case

$$\mu_n \leqq 4\{\sin^2 [1/(4n)]\}^{-1/m}\epsilon_n + 4\epsilon_n. \tag{22}$$

Since $\sin x > (2/\pi)x$ when $0 < x \leqq \pi/2$, $\sin^2[1/(4n)] \geqq n^{-2}/(4\pi^2)$. The upper bound (22) for μ_n then assumes the form

$$\mu_n \leqq Cn^{2/m}\epsilon_n + 4\epsilon_n,$$

where C depends only on m, a constant.

Finally, then,

$$|F(\theta, \phi) - Y_n(\theta, \phi)| \leqq |r_n| + |\rho_n| \leqq Cn^{2/m}\epsilon_n + 5\epsilon_n,$$

and one can state the following theorem:

THEOREM XII. *If $F(\theta, \phi)$ can be approximated by a surface spherical harmonic sum $y_n(\theta, \phi)$ of type (19) with maximum absolute error ϵ_n then the approximating sum $Y_n(\theta, \phi)$ corresponding to exponent $m>1$ represents $F(\theta, \phi)$ with maximum absolute error not exceeding*

$$Cn^{2/m}\epsilon_n + 5\epsilon_n$$

where C depends only on m.

Theorems on the convergence of $Y_n(\theta, \phi)$ to $F(\theta, \phi)$ can now be written as corollaries to Theorem XII. By Gronwall's theorem of §8, ϵ_n may be replaced by a constant multiple of $\omega(1/n)$ if $F(\theta, \phi)$ is continuous with modulus of continuity $\omega(\delta)$ on the sphere. The immediate consequence is

COROLLARY I. *If $m>2$ and if $F(\theta, \phi)$ is such that $\lim_{\delta \to 0}\omega(\delta)/\delta^{2/m}=0$, then $Y_n(\theta, \phi)$ converges uniformly to $F(\theta, \phi)$.*

If $F(\theta, \phi)$ satisfies the hypotheses of Theorem X, ϵ_n may be replaced by a constant multiple of $(1/n)\ \omega(1/n)$, and one can state

COROLLARY II. *If $m>1$ and if $F(\theta, \phi)$ has at all points of the sphere continuous first-order directional derivatives as in Theorem X, with modulus of continuity not exceeding $\omega(\delta)$, where $\lim_{\delta \to 0}\omega(\delta)/\delta^{(2/m)-1}=0$, then $Y_n(\theta, \phi)$ converges uniformly to $F(\theta, \phi)$.*

As in the case of trigonometric and polynomial approximation it is to be emphasized that the conditions imposed on $F(\theta, \phi)$ in the above corollaries are by no means necessary for the uniform convergence to $F(\theta, \phi)$ of the approximating sum corresponding to exponent m. The following observation will suffice to bring out this fact. In §9 it was pointed out that the partial sum of Laplace's series is the approximating sum corresponding to exponent $m=2$. It has already been shown in the corollary to Theorem VIII that a sufficient condition for the uniform convergence of this sum to $F(\theta, \phi)$ is that $F(\theta, \phi)$ have a modulus of continuity such that $\omega(\delta)/\delta^{1/2} \to 0$. This is a less restrictive condition on $F(\theta, \phi)$ than that afforded by Corollary II of Theorem XII. Also as in the case of other forms of approximating functions the problem treated here can be generalized by admitting a positive, continuous weight function in the integrand of (21).

Since methods are not available for finding ϵ_n by arbitrary sums other than those used in the above corollaries, the values of ϵ_n must be taken as the errors assigned in Theorems VIII and IX for representation by partial sums of Laplace's series. Conclusions arrived at by such considerations appear to be of secondary interest, and will not be included in this discussion. Fur-

thermore, since theorems on convergence for $m \leqq 1$ which employ such values of ϵ_n, and theorems on the degree of convergence for all positive values of m, are adaptations of the corresponding existing theorems for trigonometric approximation paralleling the adaptation herein given for the case $m > 1$, they, also, will not be included.

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